SPHINX—Software for Synthesizing Spherical 4R Mechanisms

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Abstract

In this paper we present SPHINX, an interactive graphics based software package for designing spherical $4R$ mechanisms. The program provides a platform for the synthesis of a mechanism that guides a body through four orientations in space. The designer is also given the ability to perform static and dynamic analyses of the resulting mechanism.

The purpose of this work is to assemble the current spherical $4R$ synthesis theory into a software package that is useful for spatial mechanism design.

1 Introduction

In spherical mechanisms, see Fig. 1, "any point in a moving body is confined to move within a spherical surface, and all spherical surfaces of motion are concentric", (Chiang, 1988). Therefore, each link of a spherical mechanisms has at most three rotational degrees of freedom. A spherical mechanism is the simplest mechanism that provides spatial movement.

Each link of a general spatial mechanism can possess up to six degrees of freedom; three rotational and three translational. Therefore, the design of spherical mechanisms is simpler than general spatial mechanisms and parallels the traditional techniques of planar mechanism synthesis.

The synthesis of planar mechanisms is inherently a two dimensional problem. Therefore, the design techniques are well suited to a drafting table, blackboard, etc. This is not true of spatial and spherical mechanisms. The inherent spatial characteristics of these mechanisms makes such two dimensional graphical constructions difficult. For these mechanisms it is useful for the designer to be able to visualize the entire problem in its three dimensions. Modern computer workstations provide the high speed graphics capabilities which make possible real-time visualization of spatial mechanisms. SPHINX uses the three dimensional graphics capabilities of an IRIS 4D-85 GT to provide the interactive environment needed to design spherical $4R$ mechanisms.

2 Overview

SPHINX is a computer graphics based interactive program for designing spherical mechanisms formed by a closed chain consisting of four revolute joints; the so called $4R$ mechanism. The result of the design is a one degree of freedom mechanism which guides a body through four finitely separated orientations in space.

The theory for the design of spherical mechanisms for four position rigid body guidance is analogous to that for planar mechanisms, see Duffy, 1980. In the planar case the designer specifies four positions in the plane and computes the set of points in the moving body which have four positions on a circle, see Sandor and Erdman, 1984. These points are the moving pivots of the planar $4R$ mechanism, and they form a cubic curve called the circle point curve. The points that are the centers of these circles are the fixed pivots of the linkage they form a cubic curve called the center point curve. See Bottema and Roth, 1979, and Suh and Radcliffe, 1978, for a further discussion of these curves.

In the case of spherical four position synthesis the designer specifies four orientations which are displayed as positions on the surface of a sphere. Let an axis in the moving body be defined as a line through the center of the sphere and a point in the body. The set of axes in the moving body which have four positions on a right circular cone are the moving axes of the spherical $4R$ mechanism and they form a cubic cone called the circle axis cone. The axes that are the central axes of these cones are the fixed axes and they form a cubic cone called the center axis cone.

The major difference between planar and spherical finite position synthesis is the essential three dimensionality of spherical mechanisms. While synthesis curves for planar mechanisms may be sketched or plotted in two dimensions, the synthesis cones for spherical mechanisms, and the mechanisms themselves, must be viewed in their full three dimensional form. The designer requires the ability to manipulate the synthesis cones and view them in an arbitrary orientation in order to gain an understanding of the available choices of fixed and moving axes. Furthermore, once the linkage has been specified the evaluation of its motion also

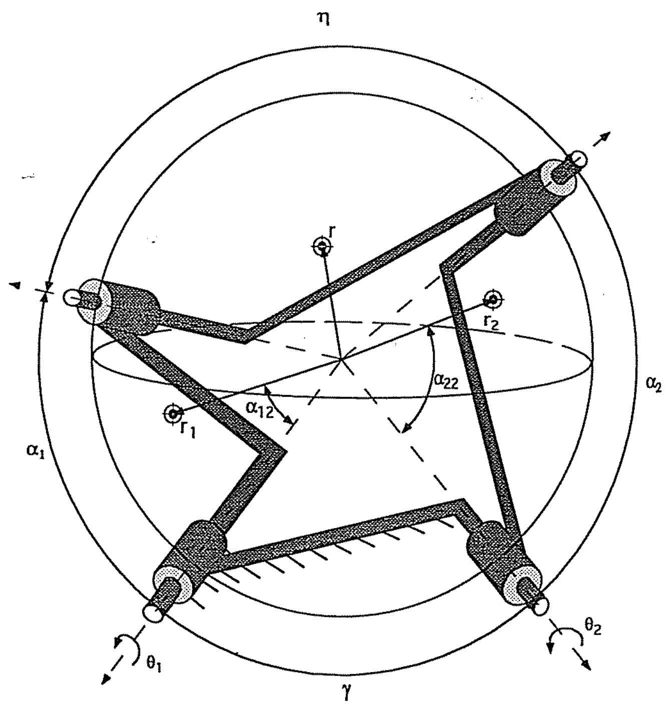

Figure 1: A Spherical 4R Mechanism

requires the ability to view the linkage in a three dimensional environment. The designer often must adjust the radii of the links and their form in order to avoid interference. Moreover, a static and dynamic analysis of the mechanisms assists in evaluating the functionality of a design. *Sphinx* provides all of these capabilities.

3 Structure

Sphinx is structured in such a manner as to provide an interactive platform from which the designer creates linkages in order to satisfy given design specifications. The program is organized such that each phase of the design process is carried out in a separate graphics window. Each window has a title bar which denotes the name of the window. Corresponding to each window is a button listed on the left hand side of the screen labeled with the name of the window. To select a window to work in use the mouse to push the appropriate button at the left hand side of the screen. The menu associated with that graphics window is then displayed on the right hand side of the screen.

4 Design Cone Generation

The design cones for four position synthesis of spherical 4R closed chain mechanisms are called the center axis and circle axis cones. They are used in the same way as the planar linkage design curves; the center point and circle point curves. For four precision position spherical synthesis the center axis cone is the set of axes that will serve as fixed axes and the circle axis cone is the set of corresponding moving axes. The procedure we use to generate these cones is located in Bodduluri et al (1991). A summary of that process follows.

In general, the quaternions that define the positions of the coupler of a 4R spherical closed chain form a curve in four dimensional space called the *image curve* of the linkage, see Ge and McCarthy, 1988. An opposite screw quadrilateral associated with four precision positions is treated as a linkage, often referred to as the *compatibility linkage*, and its image curve is computed. The center axis and circle axis cones are obtained as a projection of this curve.

The image curve of a spherical 4R linkage is the set of unit quaternions, or Euler parameters, that define the position of the coupler link relative to the base link of the compatibility linkage. The components of a quaternion are obtained from the unit vector $\mathsf{s} = (s_x, s_y, s_z)$ defining the axis of spatial rotation and θ the angle of rotation about that axis. The quaternion associated with this displacement is given by (Hamilton 1969):

$$\mathsf{S} = \left\{ \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \right\} = \left\{ \begin{array}{c} s_x \sin(\theta/2) \\ s_y \sin(\theta/2) \\ s_z \sin(\theta/2) \\ \cos(\theta/2) \end{array} \right\}. \qquad (1)$$

The image curve of the coupler of a spherical linkage is found by attaching reference frames G and H to the base and coupler of the linkage. The position of H relative to G is defined by specifying any relative angle between links. By defining the driving crank angle as the degree of freedom specified to locate the entire position of the linkage, the position of the coupler becomes parameterized by this angle. The image curve is simply this relationship expressed in quaternions. Through a linear transformation and a projection operation the rotation curve of the opposite screw quadrilateral is determined. The rotation curve of the opposite screw quadrilateral (with a reference position specified by the initial configuration of the linkage) is the center axis cone.

The method for generating the center axis cone for four positions can be used to generate the circle axis cone by inverting the relationship between the four precision positions and the fixed frame. This is done by holding a precision position fixed and considering the positions of the fixed frame relative to the precision position as it moves through the four positions. The center axis cone of the corresponding (inverted) opposite pole quadrilateral is the desired circle axis cone. These cones outline all possible linkages that pass through the desired four positions.

There is a one-to-one correspondence between axes on the center axis cone and axes on the circle axis cone. Therefore, choosing one axis on either of the design cones specifies an entire crank. Picking two pairs of axes specifies both cranks and the 4R closed chain is completely specified. The two axes on the center axis cone are connected to establish the fixed link. The two axes on the circle axis cone are joined to establish the coupler.

In SPHINX, after the four precision positions are specified and the cones created, the selection of the cranks of the mechanism is accomplished in the cones window. By choosing the appropriate button, both the driven crank and the driving crank are selected from the cones. Recall that it is necessary to choose both cranks to completely specify a linkage. After both cranks have been chosen the analyze button is used to create the corresponding linkage, which is then displayed in the linkage window.

5 Statics Module

Experience with the construction of spherical mechanisms shows that mechanisms often jam due to link deformation under internal loading. Designing links which can support these internal forces is central to synthesizing functional spherical mechanisms. The statics module of SPHINX performs a complete static loading analysis of spherical $4R$ mechanisms.

5.1 Equilibrium Equations

In this section we present the static equations of equilibrium for a rigid link i connecting two revolute joints in a spherical closed chain. The complete derivation and presentation of the static analysis algorithm used in SPHINX can be found in Larochelle and McCarthy, 1992. First, the general spatial force and moment balance equations are shown. Finally, the constraint equations associated with the requirement of spherical chains that the joint axes intersect at a point are presented.

In our formulation all of the link forces are measured in the fixed reference frame, see Fig. 2. Therefore, the force balance equation for link i is simply,

$$-{}^{0}\mathbf{f}_i + {}^{0}\mathbf{f}_{i+1} = 0. \quad (2)$$

The superscript 0 indicates that the vector is measured in the fixed frame and the subscript i denotes that the force is applied by the $(i-1)^{th}$ link to the i^{th} link.

The three moment balance equations for a general link are,

$$\mathbf{m}_i = [M_{LS}]^T \begin{Bmatrix} {}^{0}\mathbf{f}_{i+1} \\ \mathbf{m}_{i+1}, \end{Bmatrix} \quad (3)$$

where,

$$[M_{LS}] = \begin{bmatrix} [P_i]^T\,[{}^{0}T_i] \\ [T_{i+1}]^T \end{bmatrix}. \quad (4)$$

$[M_{LS}]$, the link moment matrix, is a 6×3 matrix derived from the geometry of the link.

For a spherical link i the constraint equations are as follows. Due to the statically indeterminate geometry of the joints no force transmission is allowed from link $i-1$ to link i along the i^{th} joint axis,

$${}^{0}\mathbf{f}_i \cdot {}^{0}\mathbf{z}_i = 0. \quad (5)$$

Furthermore, if there is no externally applied torque on the i^{th} joint axis,

$$\mathbf{m}_i \cdot {}^{0}\mathbf{z}_1 = 0. \quad (6)$$

Eq. 3 and Eq. 2 are the 6 equations of static equilibrium for a spherical link. These six equations written for each link of a spherical mechanism, coupled with the constraint equations, Eq. 5 and Eq. 6, form the system of linear equations to be solved. These equations are written in the form $[A]\mathbf{x} = \mathbf{b}$, where; $\mathbf{x}$ is the vector of unknown forces and moments, and $[A]$ and $\mathbf{b}$ are coefficients determined from the linear equations of static equilibrium. This system of linear equations is solved for $\mathbf{x}$ in SPHINX by using the maximum pivot strategy.

5.2 Numerical Example

In this section we present an example of the static analysis performed by SPHINX.

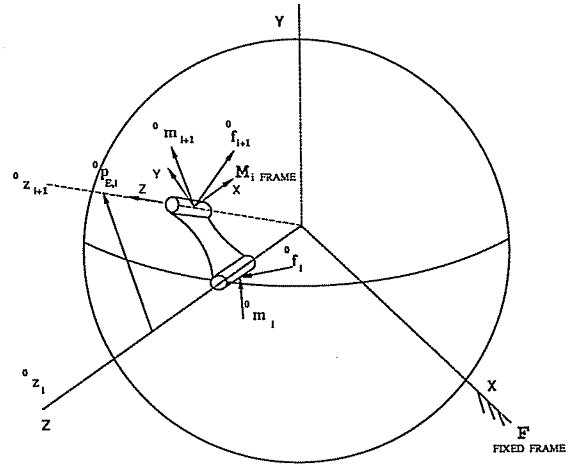

Figure 2: Free-Body Diagram of a Spherical Link

The link lengths of the mechanism analyzed are give in Tbl. 1. This is a crank-rocker mechanism. The driving torque was equal to 10.0 (Nm). The data generated by the statics module is plotted such that the internal moments and the driven torque are normalized with respect to the driving torque and are shown for the range of admissible driving crank angles.

The driven torque peaks at a driving crank angle of 264 deg. This is a singular configuration for the mechanism. The non-normalized driven torque is plotted in Fig. 3. The internal moments applied to the coupler are, $-m_{3x}$, which is the bending moment, and $-m_{3y}$, which we call the torsional moment. Recall that these moments are measured in the coupler's reference frame, see Fig. 2. The non-normalized bending and torsional moments are plotted in Fig. 4. Note that m_{3x} (solid) has a minimum value that is 1.03 times the driving torque. In addition, m_{3y} (dotted) is zero.

6 Dynamics Module

The dynamics equations are necessary for the satisfactory design of spherical mechanisms, which may satisfy kinematic specifications, but still be dynamically unsound. For instance, a linkage designed through position synthesis may reach all desired positions but require large torques to go through regions that are near a singularity. The complete derivation of the dynamic analysis of spherical $4R$ linkages can be found in Dooley and McCarthy (1992).

Our derivation uses six generalized coordinates: the driving and driven crank angles (see Fig. 1) and the Euler parameters of the coupler. The equations of motion for each moving link and the constraint equations between the bodies are presented. These equations are combined to fully prescribe the motion of the $4R$ closed chain. They are integrated using the IMSL routine, *DVERK*.

Fig. 1 is a typical spherical $4R$ mechanism. The lengths of the links are measured in terms of the rotation between the two revolute joint axes attached to the two ends of the link. Thus, the length of the driving crank is considered to be α_1, the length of the driven crank is α_2, the length of the coupler is η, and the length of the fixed link (between the two base joints) is γ. The distance from the fixed joint of the driving crank to its center of mass is α_{11}. Similarly, the distance from the fixed joint of the driven crank to its center of mass is α_{21}. The center of mass for the coupler is taken to be midway between the two floating joints. The masses of the links are m_1, m_2, and m. The inertia matrices are $[J_1']$, $[J_2']$, and $[J']$. The radial distances to the centers of mass are r_1, r_2, and r.

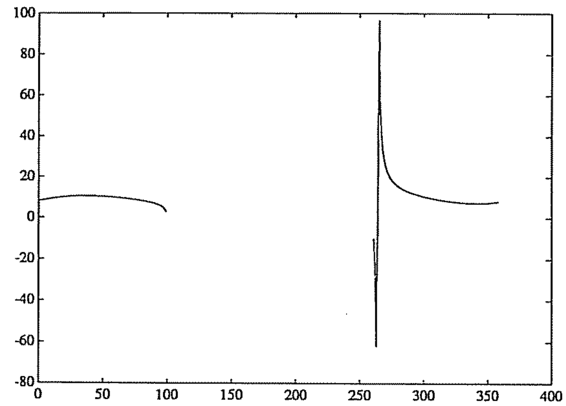

Figure 3: Driven Torque versus Driving Crank Angle

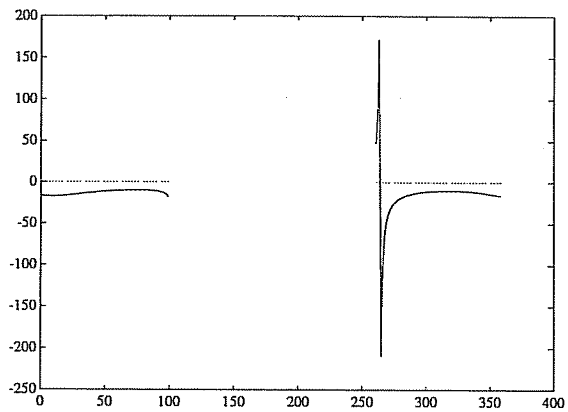

Figure 4: m_{3x} and m_{3y} versus Driving Crank Angle

The generalized coordinates used to write the equations of motion are the driving and driven crank angles and the Euler parameters defining the position of the coupler:

$$\mathbf{q} = \{\theta_1 \ \ \theta_2 \ \ Q_1 \ \ Q_2 \ \ Q_3 \ \ Q_4\}^T.$$

This system has only one degree-of-freedom. Therefore, five constraints relate the equations of motion.

6.1 Equations of Motion

Equations of motion are written individually for the three moving links of the system. The equations of motion for the driving and driven cranks are the equations for a pendulum. The equations are

$$(m_i r_i^2 \sin^2 \alpha_{i1} + J'_{ic})\ddot{\theta}_i + m_i g r_i \sin \alpha_{i1} \sin \theta_i = \Theta_i, \qquad (7)$$

where $i = 1$ for the driving crank and $i = 2$ for the driven crank, and where Θ_1 is the driving torque and $\Theta_2 = 0$. The equations of motion for the coupler are derived in terms of quaternions.

This set of generalized coordinates requires five constraint equations. One of them is the Euler parameter constraint,

$$Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 - 1 = 0.$$

Two constraints are found by noting that the driving and driven cranks have constant length. The two remaining constraints are found by recovering the driving and driven crank angles from the quaternion coordinates.

Assemble the individual equations of motion and the second derivatives of the constraints in a system of eleven equations as

$$\begin{bmatrix} M & \Gamma^T \\ \Gamma & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}+\mathbf{h} \\ \mathbf{c} \end{Bmatrix}, \qquad (8)$$

where λ is a 5-vector of LaGrange multipliers, $\mathbf{h}$ combines the Coriolis, centripetal, gravitational and applied forces, $[M]$ is the mass matrix, $[\Gamma]$ is the constraint jacobian, and $\mathbf{c}$ is a 5-vector of the product terms that result from the second derivative of the constraints.

6.2 Numerical Example

As an example of the implementation of the dynamics module we examine the forward and inverse dynamics of the mechanism detailed in Tbl. 1. The forward dynamics are integrated for a

LINK	r_i	α_i	α_{i1}	m_i	J_x	J_y	J_z
DRIVING	1.1	103.0	51.5	1.97	1.80	0.90	0.90
DRIVEN	1.3	114.3	57.1	1.83	1.56	0.78	0.78
COUPLER	1.2	16.4	8.2	1.17	1.15	0.57	0.57
FIXED	1.0	24.7	12.3				

Table 1: Properties of the $4R$ Spherical Mechanism

constant driving torque of 5.0 (Nm). The results are shown in Fig. 5 where the driving (solid) and driven (dotted) crank angles are plotted as functions of time. The inverse dynamics are calculated for a constant angular velocity of 5 (*rad/sec*). The required driving torque (solid) and the driven crank angle (dotted) are plotted versus the driving crank angle in Fig. 6.

7 Conclusions

In this article we have presented *Sphinx*, an interactive software package for designing spherical $4R$ closed chains. Incorporated into *Sphinx* are modules for performing a complete static loading analysis and both forward and inverse dynamic analysis of spherical $4R$ mechanisms.

It is our hope that this design and analysis tool will facilitate the design and construction of spherical mechanisms to solve spatial motion problems.

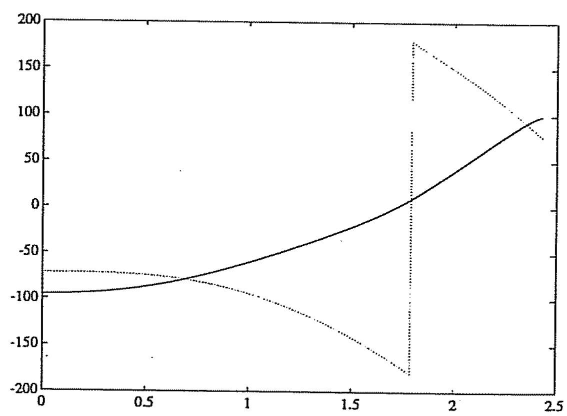

Figure 5: Driving and Driven Crank Angles versus Time

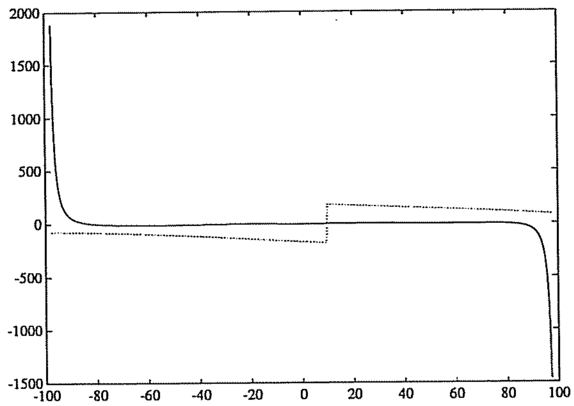

Figure 6: Driving Torque versus Driving Crank Angle

8 Acknowledgements

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A Appendix

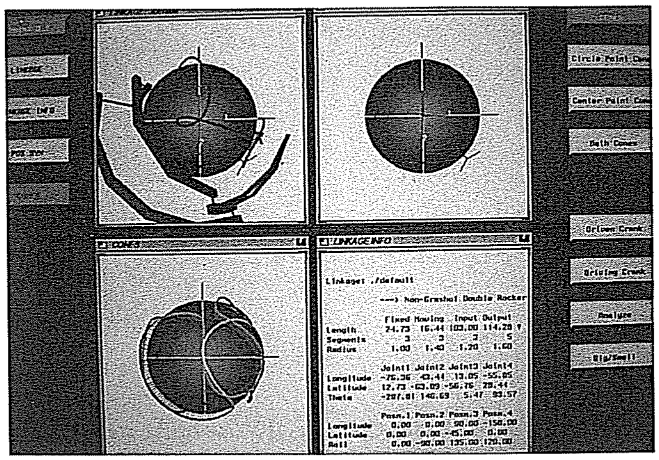

Figure 7: SPHINX

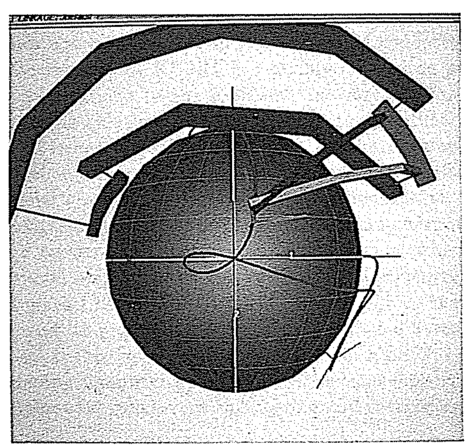

Figure 8: SPHINX: Linkage